

# Controlling of clock synchronization in WSNs: structure of optimal solutions

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August 6, 2014

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## Abstract

Energy-saving optimization is very important for various engineering problems related to modern distributed systems. We consider here a control problem for a wireless sensor network with a single time server node and a large number of client nodes. The problem is to minimize a functional which accumulates clock synchronization errors in the clients nodes and the energy consumption of the server over some time interval  $[0, T]$ . The control function  $u = u(t)$ ,  $0 \leq u(t) \leq u_1$ , corresponds to the power of the server node transmitting synchronization signals to the clients. For all possible parameter values we find the structure of optimal trajectories. We show that for sufficiently large  $u_1$  the solutions contain singular arcs.

**Keywords:** Pontryagin maximum principle, bilinear control system, singular extremals, wireless sensor network, energy-saving optimization.

## 1 Model

Power consumption, clock synchronization and optimization are very popular topics in analysis of wireless sensor networks [1]–[8]. In the majority of modern papers their authors discuss and compare communication protocols (see, for example, [5]), network architectures (for example, [4]) and technical designs by using numerical simulations or dynamical programming methods (e.g., [7]). In the present talk we consider a mathematical model related with large scale networks which nodes are equipped with noisy non-perfect clocks [2]. The task of optimal clock synchronization in such networks is reduced to the classical control problem. Its functional is based on the trade-off between energy consumption and mean-square synchronization error. This control problem demonstrates surprisingly deep connections with the theory of singular optimal solutions [9]–[14].

The network consists of a single server node (denoted by 1) and  $N$  client nodes (sensors) numbered as  $2, \dots, N + 1$ .

Let  $x_i$  be a state of the node  $i$  having the meaning of a local clock value at this node. The network evolves in time  $t \in \mathbb{R}_+$  as follows.

1) The node 1 is a time server with the perfect clock:

$$\frac{dx_1(t)}{dt} = v > 0$$

2) The client nodes are equipped with non-perfect clocks with a random Gaussian noise

$$\frac{dx_j(t)}{dt} = v + \sigma dW_j(t) + \text{synchronizing jumps},$$

where  $W_j(t)$ ,  $j = 2, \dots, N + 1$ , are independent standard Wiener processes,  $\sigma > 0$  corresponds to the strength of the noise and “synchronizing jumps” are explained below.

3) At random time moments the server node 1 sends messages to randomly chosen client nodes,  $u$  is the *intensity* of the Poissonian message flow issued from the server. The client  $j$ ,  $j = 2, \dots, N + 1$ , that receives at time  $\tau$  a message from the node 1 immediately adjusts its clock to the current value of  $x_1$ :

$$\begin{aligned} x_j(\tau + 0) &= x_1(\tau), \\ x_k(\tau + 0) &= x_k(\tau), \quad k \neq j. \end{aligned}$$

Hence the client clocks  $x_j(t)$ ,  $t \geq 0$ , are stochastic processes which interact with the time server.

The function

$$R(t) = \mathbb{E} \frac{1}{N} \sum_{j=2}^{N+1} (x_j(t) - x_1(t))^2$$

is a cumulative measure of desynchronization between the client and server nodes. Here  $\mathbb{E}$  stands for the expectation.

It was proved in [2, 3] that the function  $R(t)$  satisfies the differential equation

$$\dot{R} = -uR + N\sigma^2$$

## 2 Optimal control problem

Consider the following optimal control problem

$$\int_0^T (\alpha R(t) + \beta u(t)) dt \rightarrow \inf \tag{1}$$

$$\dot{R}(t) = -u(t)R(t) + N\sigma^2 \tag{2}$$

$$R(0) = R_0 \tag{3}$$

$$0 \leq u(t) \leq u_1 \tag{4}$$

Here  $\alpha, \beta$  are some positive constants. The control function  $u(t)$  corresponds to the power of the server node transmitting synchronization signals to the clients. The functional (1) accumulates clock synchronization errors in the clients nodes and the energy consumption of the server over some time interval  $[0, T]$ .

The admissible solutions to (1)-(4) are absolutely continuous functions, the admissible controls belong to  $L^\infty [0, T]$ .

We prove that the problem (1)-(4) has a unique solution. We find a structure of optimal control. We show that optimal solutions may contain singular arcs.

### 3 Existence of solution

**Lemma 1** *For any  $R_0$  and any parameter values  $T, \alpha, \beta, N, \sigma^2, u_1$  there exists a solution  $(\hat{R}(t), \hat{u}(t))$  to the problem (1)-(4).*

*Proof.* Let  $\mathcal{B}_{R_0}$  denote the set of continuous functions  $R : [0, T] \rightarrow \mathbb{R}$  such that  $R(0) = R_0$ . Consider the map  $K : L^\infty [0, T] \rightarrow \mathcal{B}_{R_0}$  defined as follows:

$$\begin{aligned} (Ku)(t) &= R_0 \exp \left( - \int_0^t u(\xi) d\xi \right) + N\sigma^2 \int_0^t \exp \left( - \int_s^t u(\xi) d\xi \right) ds \\ &=: A(u, t) + B(u, t). \end{aligned} \quad (5)$$

This operator assigns to the control function  $u$  the corresponding solution  $R$  of (1)-(4).

1. Let  $\{u^{(n)}(t)\}_{n=1}^\infty$  be a minimizing sequence for the functional

$$\int_0^T (\alpha R(t) + \beta u(t)) dt,$$

i.e.,

$$\int_0^T (\alpha Ku^{(n)}(t) + \beta u^{(n)}(t)) dt \rightarrow \inf_{u \in V} \left\{ \int_0^T (\alpha R(t) + \beta u(t)) dt \right\}, \quad (n \rightarrow \infty),$$

where  $V = \{v \in L^\infty [0, T] : 0 \leq v(t) \leq u_1\}$ . Recall that the space  $L^1 [0, T]$  is the adjoint space to  $L^\infty [0, T]$ . By  $\langle \phi, u \rangle$  we denote the value of the functional  $\phi \in (L^\infty [0, T])^* \cong L^1 [0, T]$  at  $u \in L^\infty [0, T]$ :

$$\langle \phi, u \rangle = \int_0^T \phi(\xi) u(\xi) d\xi.$$

Since  $u^{(n)}(t) \in [0, u_1]$ , one can extract a weakly-\* converging in  $L^\infty [0, T]$  subsequence  $u^{(n_k)}(t)$  by virtue of Banach-Alaoglu theorem. Without loss of generality one can assume that  $u^{(n)}$  weakly-\* converges to some  $\hat{u} \in L^\infty [0, T]$ . This means that for each  $\rho \in L^1 [0, T]$  one has

$$\int_0^T \rho(\xi) u^{(n)}(\xi) d\xi \rightarrow \int_0^T \rho(\xi) \hat{u}(\xi) d\xi, \quad n \rightarrow \infty. \quad (6)$$

**2.** Let us prove that the sequence  $R^{(n)}(t) := Ku^{(n)}(t)$  converges pointwise to  $\hat{R}(t) := K\hat{u}(t)$  as  $n \rightarrow \infty$ .

Further let  $\phi_s^t(\xi) := -\mathbf{1}_{[s,t]}(\xi) = \begin{cases} -1, & \xi \in [s, t], \\ 0, & \xi \notin [s, t]. \end{cases}$  Taking  $\rho(\xi) = \phi_0^t(\xi)$  in (6) we obtain

$$\int_0^t u^{(n)}(\xi) d\xi \rightarrow \int_0^t \hat{u}(\xi) d\xi, \quad n \rightarrow \infty,$$

hence

$$A(u^{(n)}, t) \rightarrow A(\hat{u}, t), \quad n \rightarrow \infty$$

for each fixed  $t$ . Note that  $B(u^{(n)}, t) = N\sigma^2 \int_0^t \exp \langle \phi_s^t, u^{(n)} \rangle ds$ . The functions  $\exp \langle \phi_s^t, u^{(n)} \rangle$  are uniformly bounded and pointwise convergent, hence Lebesgue's dominated theorem yields the convergence

$$B(u^{(n)}, t) \rightarrow B(\hat{u}, t), \quad n \rightarrow \infty$$

for each fixed  $t$ . So we established the required convergence.

**3.** Let us show that  $\hat{R}(t)$  is a solution to (1)–(4).

Obviously  $R^{(n)}(t)$  are uniformly bounded (this follows straightforward from the explicit formula (5)). Since they form a pointwise convergent sequence, Lebesgue's dominated theorem yields

$$\int_0^T \alpha R^{(n)}(t) dt \rightarrow \int_0^T \alpha \hat{R}(t) dt, \quad n \rightarrow \infty.$$

Moreover, due to weak-\* convergence, one has

$$\int_0^T \beta u^{(n)}(t) dt = \beta \int_0^T \phi_0^T(t) u^{(n)}(t) dt \rightarrow \beta \int_0^T \phi_0^T(t) \hat{u}(t) dt = \beta \int_0^T \hat{u}(t) dt.$$

This yields

$$\int_0^T (\alpha R^{(n)}(t) + \beta u^{(n)}(t)) dt \rightarrow \int_0^T (\alpha \hat{R}(t) + \beta \hat{u}(t)) dt.$$

Thus  $(\hat{R}(t), \hat{u}(t))$  is an optimal solution to (1)–(4).  $\square$

## 4 Pontryagin maximum principle

We will apply Pontryagin Maximum Principle [15] to the problem (1)–(4). Let  $(\hat{R}(t), \hat{u}(t))$  be an optimal solution. Then there exist a constant  $\lambda_0$  and a continuous function  $\psi(t)$  such that for all  $t \in (0, T)$  we have

$$H(\hat{R}(t), \psi(t), \hat{u}(t)) = \max_{0 \leq u \leq u_1} H(\hat{R}(t), \psi(t), u) \quad (7)$$

where the Hamiltonian function

$$H(R, \psi, u) = -\lambda_0(\alpha R + \beta u) + \psi(-uR + N\sigma^2)$$

Except at points of discontinuity of  $\hat{u}(t)$

$$\dot{\psi}(t) = -\frac{\partial H(\hat{R}(t), \psi(t), \hat{u}(t))}{\partial R} = \lambda_0\alpha + \hat{u}(t)\psi \quad (8)$$

And  $\psi$  satisfies the following transversality condition

$$\psi(T) = 0 \quad (9)$$

The function  $\psi(t)$  is called an adjoint function. The condition (7) is called the **maximum condition**.

The dynamics equation (2) and the adjoint equation (8) form a Hamiltonian system

$$\begin{aligned} \dot{\psi} &= \lambda_0\alpha + \hat{u}(t)\psi \\ \dot{R} &= -\hat{u}(t)R + N\sigma^2 \end{aligned} \quad (10)$$

where  $\hat{u}(t)$  satisfies the maximum condition (7). The solutions  $(R(t), \psi(t))$  of (10) are called extremals. If  $\lambda_0 \neq 0$ , we say that  $(R(t), \psi(t))$  is normal. One can show [4] that in the problem (1)-(4) every extremal is normal. So we can put  $\lambda_0 = 1$ .

## 5 Switching function and singular extremals

Denote

$$H_0(R, \psi) = -\alpha R + \psi N\sigma^2, \quad H_1(R, \psi) = -\beta - R\psi \quad (11)$$

then  $H = H_0 + uH_1$ . The Hamiltonian  $H$  is linear in  $u$ . Hence to maximize it over the interval  $u \in [0, u_1]$  we need to use boundary values depending on the sign of  $H_1$ .

$$\hat{u}(t) = \begin{cases} 0, & H_1(R(t), \psi(t)) < 0 \\ u_1, & H_1(R(t), \psi(t)) > 0 \end{cases} \quad (12)$$

The function  $H_1$  is called a switching function.

Suppose that there exists an interval  $(t_1, t_2)$  such that

$$H_1(R(t), \psi(t)) = 0, \quad \forall t \in (t_1, t_2) \quad (13)$$

then the extremal  $(R(t), \psi(t))$ ,  $t \in (t_1, t_2)$ , is called a **singular** one. In this case we can't find an optimal control from the maximum condition (7). We will differentiate the identity  $H_1(R(t), \psi(t)) \equiv 0$  by virtue of the Hamiltonian system (10) until a control  $u$  appears with a non-zero coefficient.

We say that a number  $q$  is the order of the singular trajectory iff

$$\left. \frac{\partial}{\partial u} \frac{d^k}{dt^k} \right|_{(10)} H_1(R, \psi) = 0, \quad k = 0, \dots, 2q - 1,$$

$$\left. \frac{\partial}{\partial u} \frac{d^{2q}}{dt^{2q}} \right|_{(10)} H_1(R, \psi) \neq 0$$

in some open neighborhood of the singular trajectory  $(R(t), \psi(t))$ .

It is known that  $q$  is an integer.

Singular solutions arise frequently in control problems [9]-[13] and are therefore of practical significance. We prove that for sufficiently large  $u_1$  a singular control is realised in the problem (1)-(4).

**Lemma 2** *Let*

$$\sqrt{\frac{\alpha N \sigma^2}{\beta}} \leq u_1$$

*then in the problem (1)-(4) there exists a singular extremal of order 1*

$$\hat{R}_s(t) \equiv \sqrt{N \sigma^2 \frac{\beta}{\alpha}}, \quad \psi_s(t) \equiv -\sqrt{\frac{\alpha \beta}{N \sigma^2}} \quad (14)$$

*and the corresponding singular control is*

$$u_s = \sqrt{\frac{\alpha N \sigma^2}{\beta}}$$

*Proof.* Assume that (13) holds. We will differentiate this identity along the extremal with respect to  $t$ :

$$\left. \frac{d}{dt} \right|_{(10)} H_1(R(t), \psi(t)) = 0 \quad \Rightarrow \quad -N \sigma^2 \psi(t) - \alpha R(t) = 0 \quad (15)$$

$$\left. \frac{d^2}{dt^2} \right|_{(10)} H_1(R(t), \psi(t)) = 0 \quad \Rightarrow \quad u(\alpha R(t) - N \sigma^2 \psi(t)) - 2\alpha N \sigma^2 = 0 \quad (16)$$

From (13) and (15) we have

$$R(t) = \sqrt{N \sigma^2 \frac{\beta}{\alpha}}, \quad \psi(t) = -\sqrt{\frac{\alpha \beta}{N \sigma^2}} \quad (17)$$

Substituting (17) in (16) we obtain

$$2\sqrt{N \sigma^2 \alpha \beta} \cdot u - 2\alpha N \sigma^2 = 0$$

Thus

$$R(t) \equiv \sqrt{N \sigma^2 \frac{\beta}{\alpha}}, \quad \psi(t) \equiv -\sqrt{\frac{\alpha \beta}{N \sigma^2}}$$

is a singular extremal of order 1 and  $u_s = \sqrt{\frac{\alpha N \sigma^2}{\beta}}$  is the corresponding singular control.

Note that if  $\sqrt{\frac{\alpha N \sigma^2}{\beta}} > u_1$  then  $u_s$  does not satisfy the condition  $0 \leq u(t) \leq u_1$  hence optimal solutions to the problem (1)-(4) are nonsingular.  $\square$

Recall the well-known generalized Legendre-Clebsch condition [9], the necessary condition for optimality of the singular extremal of order 1:

$$\frac{\partial}{\partial u} \frac{d^2}{dt^2} H_1(\hat{R}(t), \psi(t)) \geq 0$$

We see that this condition holds in our problem. One can show that any concatenation of the singular control with a bang control  $u = 0$  or  $u = u_1$  satisfies the necessary conditions of optimality [9].

From the transversality condition (9) it is easily seen that on the final time interval the optimal control  $\hat{u}(t)$  in the problem (1)-(4) is nonsingular. Namely, for all initial condition  $R_0$  and for all parameter values  $\alpha, \beta, N, \sigma^2, u_1$  we have the following result.

**Lemma 3** *There exists  $\varepsilon > 0$  such that  $\hat{u}(t) = 0$  for all  $t \in (T - \varepsilon, T)$ .*

*Proof.* Using the transversality condition (9) we obtain  $H_1(\hat{R}(T), \psi(T)) = -\beta < 0$ . The continuity of the switching function  $H_1$  implies that

$$H_1(\hat{R}(t), \psi(t)) < 0 \quad \forall t \in (T - \varepsilon, T)$$

for some  $\varepsilon > 0$ . The maximum condition (7) yields  $\hat{u}(t) = 0, \quad t \in (T - \varepsilon, T)$ .  $\square$

## 6 The orbits of the Pontryagin maximum principle system

Consider the behaviour of the extremals on the plane  $(R, \psi)$ . Let  $\Gamma$  be a switching curve, that is, a set of point such that  $H_1(R, \psi) = 0$ . By (11) we have  $\Gamma = \{(R, \psi) | \beta + R\psi = 0\}$ . We are interested in the domain  $\{(R, \psi) : R > 0\}$ . Denote

$$\Gamma^+ = \Gamma \cap \{(R, \psi) : R > 0\}$$

Above  $\Gamma^+$  the optimal control  $\hat{u}$  equals 0, below  $\Gamma^+$  the optimal control  $\hat{u}$  equals  $u_1$  (see (12)). Let  $u = 0$  then the Hamiltonian system (10) has the form

$$\dot{R} = N\sigma^2, \quad \dot{\psi} = \alpha \tag{18}$$

The general solution of (18) is

$$R(t) = N\sigma^2 t + C_1, \quad \psi(t) = \alpha t + C_2$$

On the plane  $(R, \psi)$  the orbits of the system (18) are straight lines

$$\psi = \frac{\alpha}{N\sigma^2} R + C_3$$

Let  $u = u_1$  than the Hamiltonian system (10) has the form

$$\dot{R} = -u_1 R + N\sigma^2, \quad \dot{\psi} = \alpha + u_1 \psi \quad (19)$$

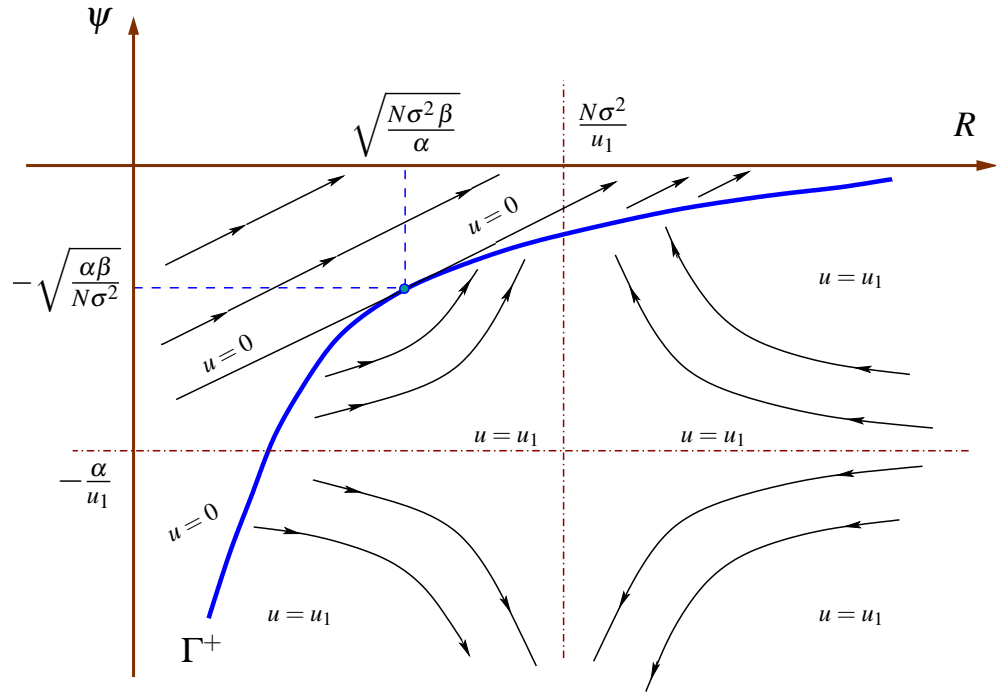
The general solution of (19) is

$$R(t) = \tilde{C}e^{-u_1 t} + \frac{N\sigma^2}{u_1}, \quad \psi(t) = \tilde{w}e^{u_1 t} - \frac{\alpha}{u_1}$$

On the plane  $(R, \psi)$  if  $\tilde{C} \neq 0$ ,  $\tilde{w} \neq 0$ , the orbits of the system (19) are hyperbolas

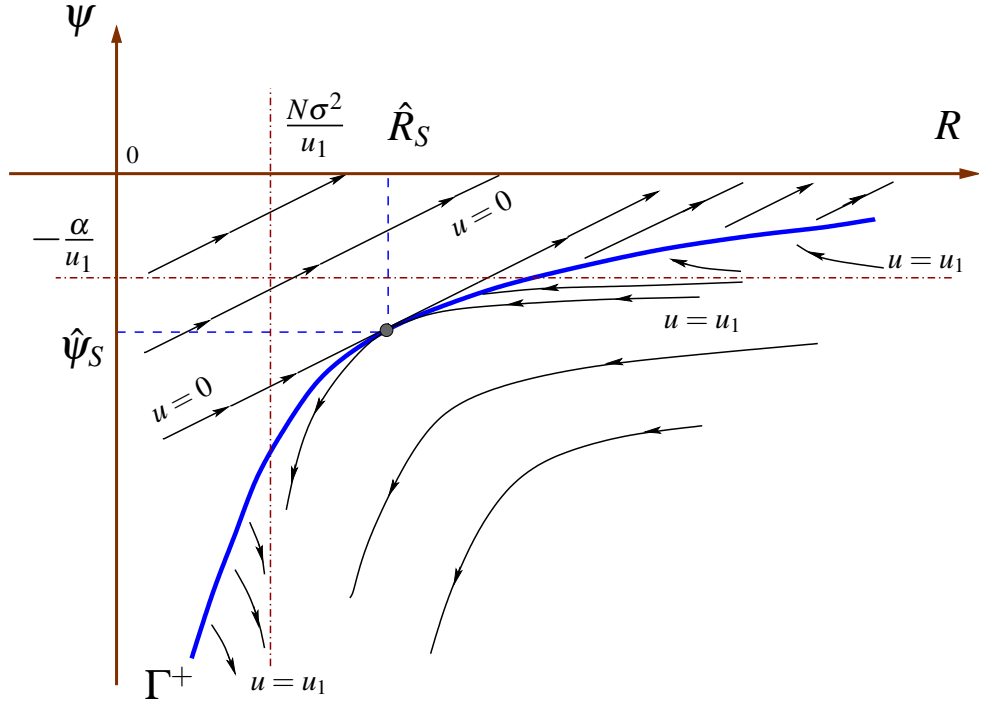
$$|\alpha + \psi u_1| \cdot |N\sigma^2 - u_1 R| = \omega$$

If  $\tilde{C} = 0$ ,  $\tilde{w} \neq 0$ , the orbit is the straight line  $R = \frac{N\sigma^2}{u_1}$ , directed upward if  $\tilde{w} > 0$  or downward if  $\tilde{w} < 0$ . If  $\tilde{w} = 0$ , the orbit is the straight line  $\psi = -\frac{\alpha}{u_1}$ , directed to the left if  $\tilde{C} > 0$  or to the right if  $\tilde{C} < 0$ . If  $\tilde{C} = 0$ ,  $\tilde{w} = 0$ , the point  $\left(\frac{N\sigma^2}{u_1}, -\frac{\alpha}{u_1}\right)$  is the stationary orbit.



**Fig 1.** Orbits in the nonsingular case:  $\sqrt{\frac{\alpha N\sigma^2}{\beta}} > u_1$





**Fig 2.** Orbits in the singular case:  $\sqrt{\frac{\alpha N \sigma^2}{\beta}} \leq u_1$

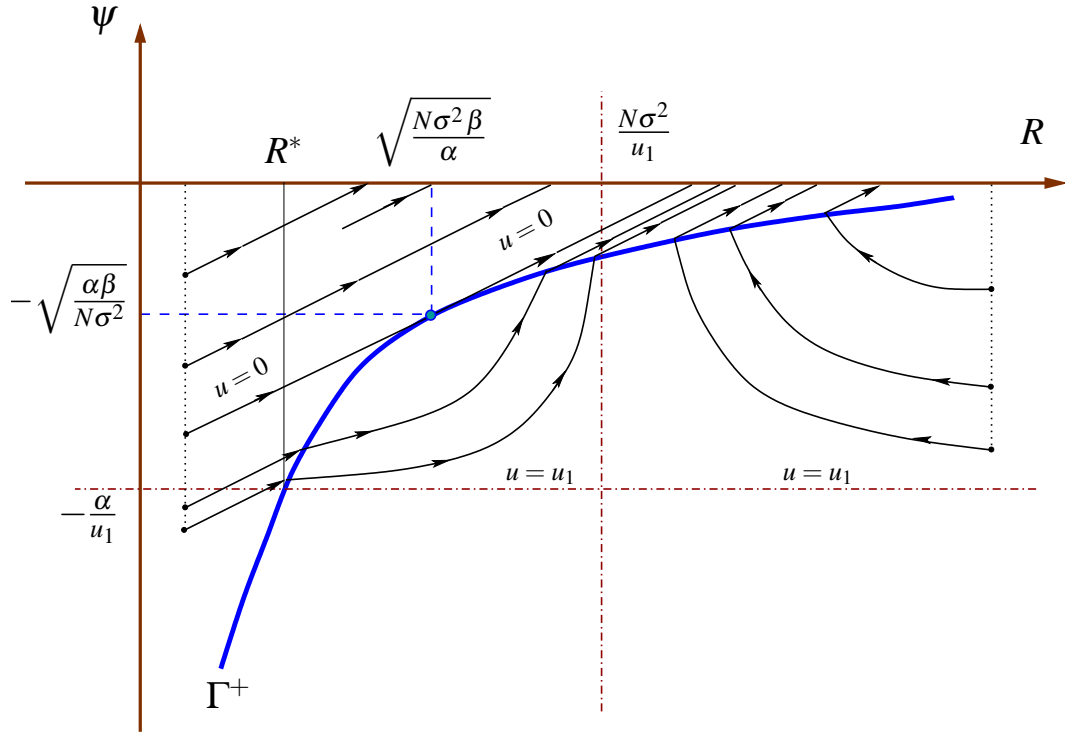
**Remark.** On these figures we don't show trajectories  $(R(t), \psi(t))$  with  $\psi(0) > 0$  because they cannot satisfy the transversality condition.

Note that in the case  $\sqrt{\alpha N \sigma^2 / \beta} \leq u_1$  two extremals go out of the singular point  $\left( \sqrt{N \sigma^2 \frac{\beta}{\alpha}}, -\sqrt{\frac{\alpha \beta}{N \sigma^2}} \right)$  (with  $u = 0$  and  $u = u_1$ ). But only one extremal (going of the singular point) satisfies the transversality condition (9).

Thus for any  $R_0 \geq 0$  there exists a unique extremal such that  $R(0) = R_0$ ,  $\psi(T) = 0$ . Since we prove that a solution to problem (1)-(4) exists hence the constructed extremals are optimal.

To summarize the above analysis in the next two sections we consider separately the nonsingular and singular cases. In each case we provide a plot with optimal solutions and state a conclusion on the structure of the optimal control  $\hat{u}(t)$  (Theorems 1 and 2). It is interesting also to see how the structure of  $\hat{u}(t)$  depends on the parameter  $R_0$  and  $T$ . The answer is presented on Figures 4 and 6.

## 7 Optimal solutions. Nonsingular case



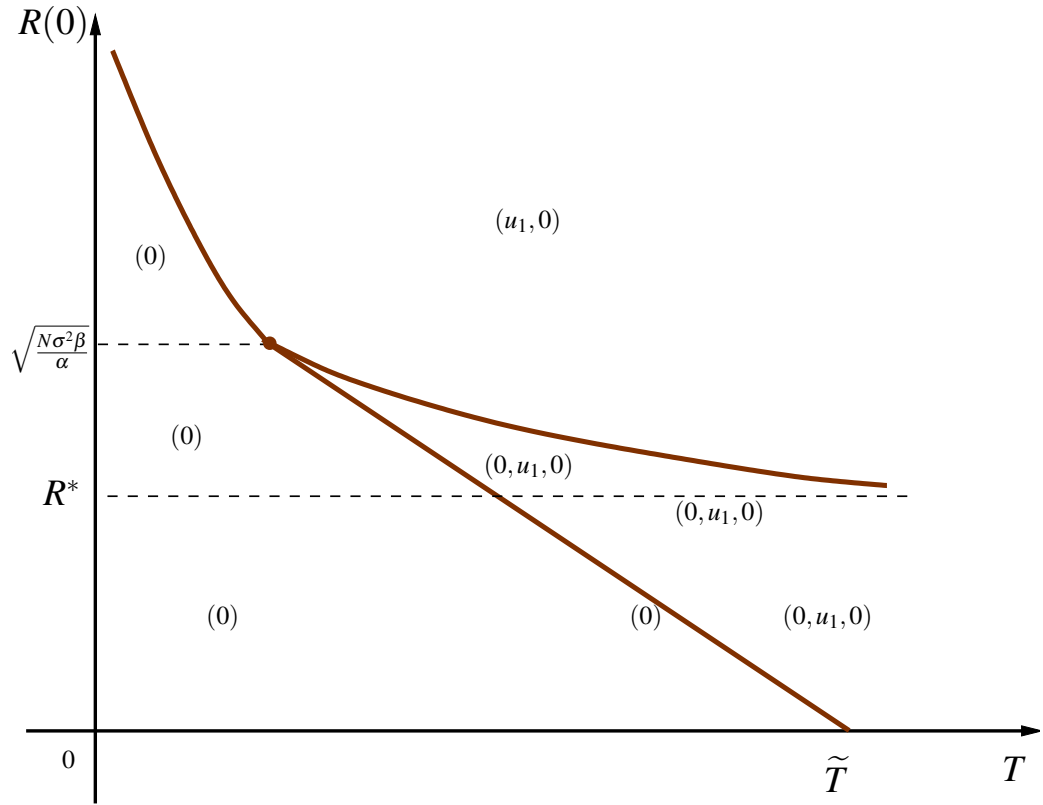
**Fig 3.** Optimal solutions for different values of the problem parameters.  
*Nonsingular case.*

**Theorem 1** Let  $\sqrt{\frac{\alpha N \sigma^2}{\beta}} > u_1$ , that is, optimal solutions are nonsingular (Lemma 2). Then, depending of values  $R(0)$  and  $T$ , the optimal control  $\hat{u}(t)$  has one of the following forms

- 1.1.  $\hat{u}(t) = 0, t \in (0, T)$
- 1.2.  $\hat{u}(t) = \begin{cases} u_1, & t \in (0, t_1) \\ 0, & t \in (t_1, T) \end{cases}$
- 1.3.  $\hat{u}(t) = \begin{cases} 0, & t \in (0, t_1) \\ u_1, & t \in (t_1, t_2) \\ 0, & t \in (t_2, T) \end{cases}$

i.e., the optimal control switches between  $u = 0$  and  $u = u_1$  and the number of switchings does not exceed 2.

The Fig. 4 shows how the *structure* of optimal controls  $\hat{u} = \hat{u}(t)$ ,  $t \in [0, T]$ , depends on  $T$  and on the initial value  $R(0)$ .



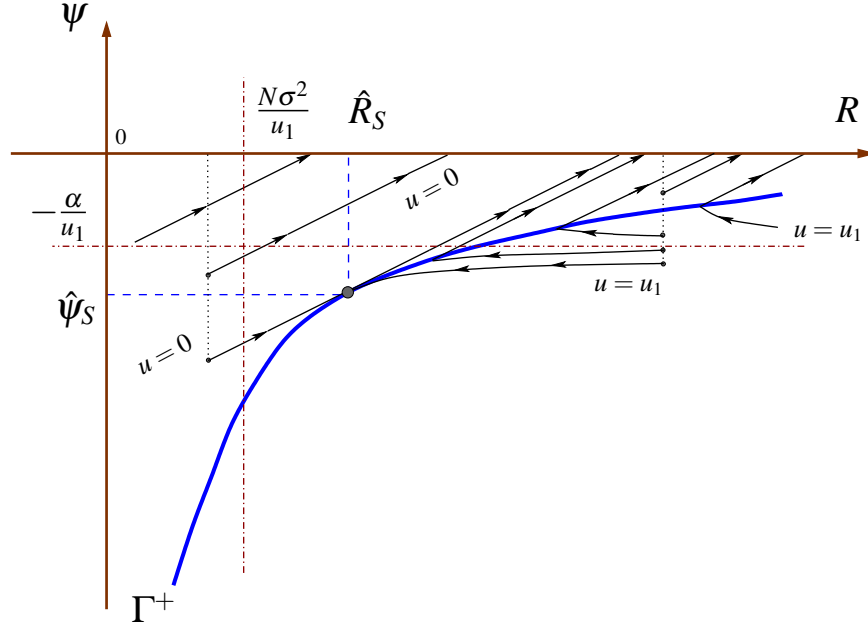
**Fig 4.**

Let  $(\theta, \rho)$  be some point on the plane  $(T, R(0))$ . Assume that  $(\theta, \rho)$  belongs to a domain labeled, *for example*, by  $(a, b, c)$ . This means that for the optimal control problem with  $T = \theta$  and  $R(0) = \rho$  the optimal control function  $\hat{u}(t)$  has the following form

$$\hat{u}(t) = \begin{cases} a, & t \in (0, \tau_1), \\ b, & t \in (\tau_1, \tau_2), \\ c, & t \in (\tau_2, \theta). \end{cases}$$

Here  $\tau_1$  and  $\tau_2$  are some numbers satisfying the condition  $0 < \tau_1 < \tau_2 < \theta$ . The numbers  $\tau_1$  and  $\tau_2$  depend on  $(\theta, \rho)$  and on all parameters  $(\alpha, \beta, N, \sigma)$  of the model. For points  $(\theta, \rho)$  in the domain labeled by  $(0)$  we have  $\hat{u}(t) = 0$  for all  $t \in [0, T]$ .

## 8 Optimal Solutions. Singular case



**Fig 5.** Optimal solutions for different values of the model parameters.  
*Singular case.*

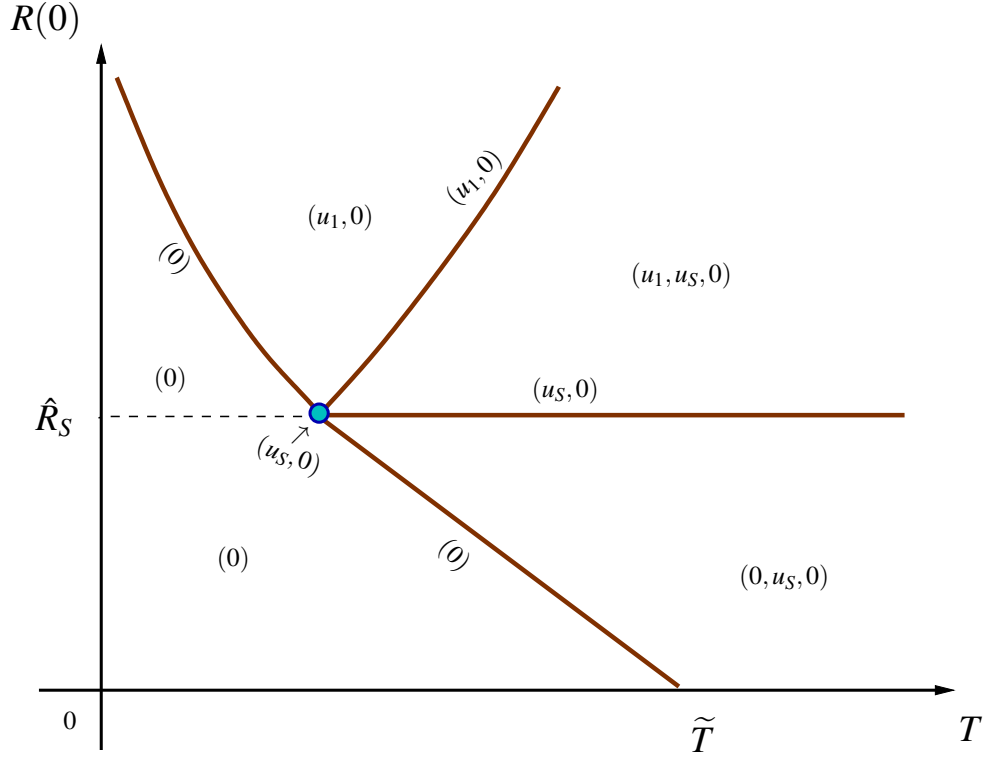
**Theorem 2** Let  $\sqrt{\frac{\alpha N \sigma^2}{\beta}} \leq u_1$ . Then, depending of values  $R(0)$  and  $T$ , the optimal control  $\hat{u}(t)$  has one of the following forms

$$2.1. \quad \hat{u}(t) = 0, \quad t \in (0, T)$$

$$2.2. \quad \hat{u}(t) = \begin{cases} u_1, & t \in (0, t_1) \\ 0, & t \in (t_1, T) \end{cases} \quad 2.3. \quad \hat{u}(t) = \begin{cases} u_s, & t \in (0, t_1) \\ 0, & t \in (t_1, T) \end{cases}$$

$$2.4. \quad \hat{u}(t) = \begin{cases} 0, & t \in (0, t_1) \\ u_s, & t \in (t_1, t_2) \\ 0, & t \in (t_2, T) \end{cases}, \quad 2.5. \quad \hat{u}(t) = \begin{cases} u_1, & t \in (0, t_1) \\ u_s, & t \in (t_1, t_2) \\ 0, & t \in (t_2, T) \end{cases}$$

*i.e., the number of control switchings does not exceed 2 and the optimal solutions may contain the singular arcs (cases 2.3-2.5).*

**Fig 6.**

As it is seen from Fig. 6 in the *singular case* on the plane  $(T, R(0))$  we have more domains with different structures of the optimal control  $\hat{u} = \hat{u}(t)$ . These additional domains are labeled as  $(u_s, 0)$  or  $(a, u_s, 0)$ . Note that on that intervals  $t \in \Delta$  where  $\hat{u}(t) = u_s$  the function  $\hat{R}(t)$  takes the constant value  $\hat{R}_s$ :

$$\hat{R}(t) = \hat{R}_s, \quad t \in \Delta.$$

## 9 Conclusions

We considered the control problem for wireless sensor networks with a single time server node and a large number of client nodes. The cost functional of this control problem accumulates clock synchronization errors in the clients nodes and the energy consumption of the server over some time interval  $[0, T]$ . For all possible parameter values we found the structure of optimal control function. It was proved that for any optimal solution  $\hat{R}(t)$  there exist a time moment  $\tau$ ,  $0 \leq \tau < T$ , such that  $\hat{u}(t) = 0$ ,  $t \in [\tau, T]$ , i.e., the sending messages at times close to  $T$  is not optimal. We showed that for sufficiently large  $u_1$  the optimal solutions contain singular arcs. We found conditions on the model parameters under which different types of the optimal control are realized.

We hope that our study of the energy-saving optimization will also be usefull for analysis of other engineering problems related to modern distributed systems. In future we plan to extend these results to more general models.

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